Instructions:

- Solve all 6 problems.
- Start each problem on a new sheet of paper.
- Define all variables that are introduced in solving the problems.
- No calculators or electronic devices are allowed.
- Justify all answers and show all work.
- Each problem carries the same weight, but individual subproblems may be weighted differently.

1. Consider the matrix

\[
A = \begin{pmatrix}
1 & 0 \\
1 & 1 \\
0 & 1
\end{pmatrix}
\]

(a) Find the singular values of \( A \).
(b) Using part (a), find the full singular value decomposition of \( A \).
(c) Using the singular value decomposition of part (b), write \( A \) as the sum of rank one matrices.

2. The first three Laguerre polynomials are: \( 1 \), \( 1 - t \), \( 2 - 4t + t^2 \).
(a) Show that these polynomials form a basis, \( B \), for polynomial space \( \mathbb{P}_2 \).
(b) Find the coordinate vector of \( p(t) = 7 - 8t + 3t^2 \) relative to this basis \( B \).
3. Prove the following Diagonalization Theorem:

An \( n \times n \) matrix \( A \) is diagonalizable if and only if \( A \) has \( n \) linearly independent eigenvectors. That is, \( A \) can be written as \( A = PDP^{-1} \), with \( D \) a diagonal matrix, if and only if \( A \) has \( n \) linearly independent eigenvectors. Moreover, the diagonal entries of \( D \) are the eigenvalues of \( A \), and the columns of \( P \) are, respectively, the corresponding eigenvectors of \( A \).

4. Prove or disprove each of the following statements:
   a) \( S = \{(x, y, z) \in \mathbb{R}^3 \mid (x^2 + y^2)(x + y) = 0\} \) is a vector subspace of \( \mathbb{R}^3 \).
   b) \( S = \{(x, y, z) \in \mathbb{C}^3 \mid (x^2 + y^2 + z^2)(x + y - z) = 0\} \) is a vector subspace of \( \mathbb{C}^3 \).

5. A real \( n \times n \) matrix \( M \) is called a Markov matrix if all entries of \( M \) are nonnegative and the sum of the entries in each column vector of \( M \) is equal to 1.

   a) Let \( x \in \mathbb{R}^n \) be a column vector with nonnegative entries and such that these entries add up to 1. Prove that if \( M \) is an \( n \times n \) Markov matrix also \( Mx \) is a column vector whose entries add up to 1.
   b) Prove that a Markov matrix always has an eigenvalue 1 and that all other eigenvalues are in absolute value smaller or equal to 1.

6. Find an orthonormal basis for the vector subspace of \( \mathbb{R}^4 \) spanned by \[ \left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \]